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Wilson Line Correlators in $\mathcal{N} = 4$ Non-commutative Gauge Theory on $S^2 \times S^2$

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Abstract

We investigate the Wilson line correlators dual to supergravity multiplets in $\mathcal{N} = 4$ non-commutative gauge theory on $S^2 \times S^2$. We find additional non-analytic contributions to the correlators due to UV/IR mixing in comparison to ordinary gauge theory. Although they are no longer BPS off shell, their renormalization effects are finite as long as they carry finite momenta. We propose a renormalization procedure to obtain local operators with no anomalous dimensions in perturbation theory. We reflect on our results from dual supergravity point of view. We show that supergravity can account for both IR and UV/IR contributions.

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1 Introduction

It is a very attractive idea to obtain string theory in the large N limit of gauge theory [1]. The most concrete proposal so far is AdS/CFT correspondence [2]. The correspondence between the BPS sectors of the both theories is well established [3][4]. The correspondence is extended further to (close to BPS) BMN operators [5] and to integrable sectors [6]∼[10]. Since the correspondence is the duality between the weak and strong coupling regimes, we presumably require non-perturbative formulation of supersymmetric gauge theories to make further progress.

In fact matrix models have been proposed to formulate string theory at the non-perturbative level [12][13]. The most attractive feature of these constructions is that they preserve supersymmetry. Non-commutative (NC) gauge theory can be realized by expanding the matrices around the flat non-commutative solutions in the large N limit [14][15][16]. A fuzzy homogeneous space G/H can be obtained as a classical solution by introducing a Myers term [17][18][19] with finite N. It may extremize the effective action of IIB matrix model at quantum level [32][34]. Although SUSY is broken softly in these models at the scale where the manifold is curved, it will not affect local properties of the theory . In this sense non-perturbative formulation of supersymmetric NC gauge theory may be realized through matrix models.

Just like ordinary gauge theory, supersymmetric NC gauge theory is expected to possess dual supergravity description [20][21]. In fact there are BPS operators in matrix models which serve as the vertex operators for supergravity multiplets [22][23]. In a non-commutative spacetime, they reduce to a special type of the Wilson lines which are the only gauge invariant operators in NC gauge theory [24]. However they are no longer BPS operators off-shell unlike ordinary gauge theory. In fact they are renormalized in general and the justification of supergravity description requires more work than ordinary gauge theory. They are referred to as SUGRA operators in this paper. The main purpose of this paper is to advance our understandings in this issue.

We investigate non-commutative extensions of chiral operators in $\mathcal{N}=4$ NC gauge theory which is realized on fuzzy $S^2 \times S^2$ in the large N limit. Since compact spaces can be realized in matrix models with finite N, such a construction may enable us to understand non-perturbative (finite N) effects in string theory. We first construct these operators in the matrix model in section 2. We subsequently investigate the correlators of them in

perturbation theory in section 2 and 3. The two point correlators receives two types of contributions. The first type is positive definite just like ordinary gauge theory while the second type contains non-commutative phases. We call them planar and non-planar contributions respectively in this paper. In the literature, it has been often assumed that the correlators reduce to those of ordinary gauge theory in the small momentum limit. However that needs to be examined due to UV/IR mixing effects [25]. In fact we have found that the correlators receive UV/IR contributions in addition to IR contributions which are the sole contributions in ordinary gauge theory [1].

Non-planar contributions also play an important role to understand the renormalization property of the SUGRA operators. Since NC gauge theory is not a local field theory, the introduction of the notion of locality requires a considerable work at quantum level. We find in section 3 that the Wilson line correlators receive logarithmic corrections at the one loop level. We recall here that the Wilson lines with different momenta are different operators and there is a freedom to rescale them by momentum dependent factors. Since the renormalization effect can be associated with each Wilson line, we propose a perturbative prescription to rescale the operators in such a way that they can be interpreted as the Fourier transformation of the local operators with no anomalous dimensions.

Such a prescription is certainly necessary to make contact with supergravity. We investigate dual supergravity description of the correlators in section 4. The precise prescription to apply such a correspondence is not fully understood such as where to locate the Wilson lines in the fifth radial coordinate. In this paper we study SUGRA operators in detail in order to understand this problem. We show that we can successfully reproduce the essential features of the correlators in NC gauge theory by locating the Wilson lines at the maximum of string metric as proposed in [30]. We conclude in section 5 with discussions.

2 Wilson lines on homogeneous spaces

NC gauge theory on homogeneous space G/H has been investigated in our recent work [31]∼[35]. In this construction, a fuzzy spacetime is represented by p_μ which is a Lie algebra of G. p_μ and gauge field a_μ are unified into the basic matrix degrees of freedom as $A_\mu = f(p_\mu + a_\mu)$. f is the coefficient of a Myers term. The 't Hooft coupling is identified as $\lambda^2 \sim n^2/f^4N$ in 2 and 4 dimensional NC gauge theories with $U(n)$ gauge group. We have

[1] The large momentum limit of the Wilson line correlators has been investigated in [27][28][29].

clarified the large N scaling behavior of the theory by power counting arguments with fixed 't Hooft couplings [32]. These predictions for the large N scaling behavior of matrix models are confirmed by recent non-perturbative investigations [37][38][39].

In this section, we first construct the gauge invariant observables, namely the Wilson lines in NC gauge theory on G/H. They are the single traced object made of polynomials of matrices. The structure of these observables is dictated by the isometry G. On S^2, we consider the following polynomials of A_μ

$$y_{jm}^{\alpha_1,\alpha_2,\cdots,\alpha_j} Tr A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_j} A_{i_1} A_{i_2} \cdots A_{i_k}, \tag{2.1}$$

where $\alpha = 0, 1, 2$ denote the dimensions in which S^2 extends. $y_{jm}^{\alpha_1,\alpha_2,\cdots,\alpha_j}$ denotes a totally symmetric traceless tensor which corresponds to the spin j representation of $SU(2)$. The one point functions of these observables vanish since they carry non-vanishing angular momentum. On $S^2 \times S^2$, we can construct analogous operators

$$y_{jm}^{\alpha_1,\alpha_2,\cdots,\alpha_j} y_{pq}^{\beta_1,\beta_2,\cdots,\beta_p} Tr A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_j} A_{\beta_1} A_{\beta_2} \cdots A_{\beta_p} A_{i_1} A_{i_2} \cdots A_{i_k}, \tag{2.2}$$

where $\beta = 3, 4, 5$ denote the dimensions in which the second S^2 extends.

Since $i = 6 \sim 9$, these operators are classified by the representations of $SO(4)$ or its subgroup. The simplest operators of this kind possess the $U(1)$(R) charge which is equal to the number of the Z fields in the operator:

$$y_{jm}^{\alpha_1,\alpha_2,\cdots,\alpha_j} y_{pq}^{\beta_1,\beta_2,\cdots,\beta_p} Tr A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_j} A_{\beta_1} A_{\beta_2} \cdots A_{\beta_p} Z^J, \tag{2.3}$$

where $Z = (A_8 + iA_9)/\sqrt{2}$. Although this operator is the analogue of the chiral operator in ordinary gauge theory, it is not invariant under SUSY transformation of IIB matrix model. It is due to the presence of gauge fields in addition to Z^J which carry definite momenta. In contrast we do not need such structure in ordinary gauge theory. Therefore this operator will be renormalized in generic cases. The main goal of this paper is to understand the renormalization property of SUGRA operators in the small momentum regime.

In the case of S^2, the background p_μ consists of angular momentum operators in the spin l representation. In this paper we focus on a simple 4d manifold $S^2 \times S^2$ where the both S^2 are of the identical size: $l_1 = l_2 = l$ and $N = n(2l+1)^2$ with $U(n)$ gauge group. In our expansion of A_μ around a background p_μ, the leading term of the Wilson line is:

$$f^{j+p+J} y_{jm}^{\alpha_1,\alpha_2,\cdots,\alpha_j} y_{pq}^{\beta_1,\beta_2,\cdots,\beta_p} Tr p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_j} p_{\beta_1} p_{\beta_2} \cdots p_{\beta_p} z^J, \tag{2.4}$$

where $z = (a_8 + ia_9)/\sqrt{2}$. Without the loss of generality, we can focus on the highest weight states of $SU(2) \times SU(2)$:

$$y_{j,j} y_{p,p} Tr(p_+)^j (\tilde{p}_+)^p z^J = Tr\mathcal{Y}_{j,p} z^J, \tag{2.5}$$

where we have also rescaled the operator to absorb the factor of f^{j+p+J}.

Since we normalize $Tr\mathcal{Y}^\dagger_{j,p}\mathcal{Y}_{j,p} = n$, the coefficient $y_{j,j}$ is determined semiclassically:

$$tr(p_+)^j(p_-)^j \sim (\frac{1}{2})^j tr(p_1^2 + p_2^2)^j \sim (\frac{l^2}{2})^j l \int dcos\theta sin^{2j}\theta \tag{2.6}$$

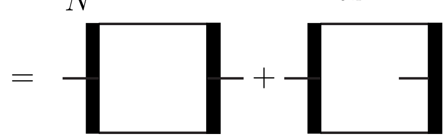
as

$$y_{j,j}^2 = \frac{(2j+1)!!}{(2j)!!}\Big(\frac{2}{l^2}\Big)^j \frac{1}{2l}. \tag{2.7}$$

If we replace one of p_μ by a gauge field a_μ, we essentially obtain an operator with a different spherical harmonics $\mathcal{Y}_{j-1,p}$. Since $y_{j,j} \sim l^{-(j+1/2)}$, such an operator is suppressed by $1/l$ in comparison to the original one. Thus the non-leading terms which contain gauge fields instead of p_μ are suppressed by powers of $1/l$ in comparison to the leading term. We therefore neglect them in the subsequent investigations. We will indeed find in section 3 that they can be neglected in the large N limit with a fixed 't Hooft coupling λ^2 as long as j, p are $O(1)$. However their effect becomes important when $j^2 + p^2 \sim l/\lambda$. It is expected the case since the momenta approaches the non-commutative scale. In this paper, we assume that momenta j, p of the Wilson lines are much smaller than the non-commutative scale .

We first investigate the two point functions of $Tr\mathcal{Y}_{j,p}z^2$ with $J = 2$ which correspond to the most relevant chiral operator in ordinary $SU(n)$ gauge theory. At the tree level, we obtain both the planar and non-planar contributions as follows

$$\frac{n}{N} < Tr\mathcal{Y}_{j,p} z^2 Tr\bar{z}^2 \mathcal{Y}^\dagger_{j,p} >$$



$$= \ < j,p|\frac{1}{P_2^2 P_3^2}|j,p>_p + < j,p|\frac{1}{P_2^2 P_3^2}|j,p>_{np}, \tag{2.8}$$

where

$$\begin{aligned} P_i^\mu \mathcal{Y}_{j_{i'} m_{i'} p_{i'} q_{i'}} &\equiv [p^\mu, \mathcal{Y}_{j_{i'} m_{i'} p_{i'} q_{i'}}]\delta_{ii'}, \\ \mathcal{Y}_{jmpq} &\equiv y_{jm}^{\alpha_1,\alpha_2,\cdots,\alpha_j} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_j} y_{pq}^{\beta_1,\beta_2,\cdots,\beta_p} p_{\beta_1} p_{\beta_2} \cdots p_{\beta_p}. \end{aligned} \tag{2.9}$$

We have also introduced the following average:

$$
\begin{aligned}
< j,p|X|j,p>_p &= \frac{n^3}{f^8N}\sum_{j_2,j_3,m_2,m_3}\sum_{p_2,p_3,q_2,q_3}\Psi^*_{123}X\Psi_{123},\\
< j,p|X|j,p>_{np} &= \frac{n^3}{f^8N}\sum_{j_2,j_3,m_2,m_3}\sum_{p_2,p_3,q_2,q_3}\Psi^*_{132}X\Psi_{123},\\
\Psi_{123} &\equiv Tr\mathcal{Y}_{j_3m_3p_3q_3}\mathcal{Y}_{j_2m_2p_2q_2}\mathcal{Y}_{jp}. \qquad (2.10)
\end{aligned}
$$

The planar amplitude is

$$
\begin{aligned}
< j,p|\frac{1}{P_2^2P_3^2}|j,p>_p &= \frac{n^3}{f^8N}\sum_{j_2,j_3,p_2,p_3}\frac{(2j_2+1)(2p_2+1)(2j_3+1)(2p_3+1)}{(j_2(j_2+1)+p_2(p_2+1))(j_3(j_3+1)+p_3(p_3+1))}\\
&\times\left\{\begin{array}{ccc} j & j_2 & j_3\\ l & l & l\end{array}\right\}^2\left\{\begin{array}{ccc} p & p_2 & p_3\\ l & l & l\end{array}\right\}^2, \qquad (2.11)
\end{aligned}
$$

where we refer [40] for $6j$ symbols. Apart from a numerical factor, this function is identical to $\omega(P^2)$ which appeared in the one loop self-energy of gauge fields [35]. The non-planar amplitude is

$$
\begin{aligned}
< j,p|\frac{1}{P_2^2P_3^2}|j,p>_{np} &= \frac{n^3}{f^8N}\sum_{j_2,j_3,p_2,p_3}\frac{(2j_2+1)(2p_2+1)(2j_3+1)(2p_3+1)}{(j_2(j_2+1)+p_2(p_2+1))(j_3(j_3+1)+p_3(p_3+1))}\\
&\times e^{i\phi_{123}}\left\{\begin{array}{ccc} j & j_2 & j_3\\ l & l & l\end{array}\right\}^2\left\{\begin{array}{ccc} p & p_2 & p_3\\ l & l & l\end{array}\right\}^2, \qquad (2.12)
\end{aligned}
$$

where $e^{i\phi_{123}} = (-1)^{j+j_2+j_3+p+p_2+p_3}$. Note that this amplitude is planar with respect to the gauge group indices. We investigate only planar sectors with respect to the gauge group indices by assuming n is large in this paper. Unlike large momentum regime, it cannot be neglected in comparison to the planar amplitude (2.11) in the small momentum regime. It exhibits extra non-analytic behavior with respect to the external momenta due to UV/IR mixing as we shortly demonstrate.

A detailed investigation of the planar amplitude using the Wigner approximation of $6j$ symbols have been carried out in [35]. Such an approximation can be justified for large external momenta $j^2+p^2>>1$. Here we investigate these amplitudes by using Edmonds' approximation for 6j symbols:

$$
\left\{\begin{array}{ccc} j & j_2 & j_3\\ l & l & l\end{array}\right\} = \frac{(-1)^{j_3}}{\sqrt{(2j_3+1)(2l+1)}}d^{(j)}_{j_3-j_2,0}(\theta_3), \qquad (2.13)
$$

where

$$
cos(\theta_3) = -\frac{1}{2}\sqrt{\frac{j_3(j_3+1)}{l(l+1)}}, \qquad (2.14)
$$

and

$$d^{(j)}_{m'm}(\beta) = (jm'|exp(i\beta J_y)|jm). \tag{2.15}$$

Since it is valid when $j_2, j_3 >> j$, this approximation enables us to estimate logarithmically divergent amplitudes with finite external momenta .

Under this approximation, the planar amplitude becomes

$$(\frac{n^2}{f^4N})^2 \sum_{j_2,j_3,p_2,p_3} \frac{(2j_2+1)(2p_2+1)}{(j_2(j_2+1)+p_2(p_2+1))(j_3(j_3+1)+p_3(p_3+1))} \times(d^{(j)}_{j_3-j_2,0}(\theta_3))^2(d^{(p)}_{p_3-p_2,0}(\tilde{\theta}_3))^2. \tag{2.16}$$

Since $|j_3-j_2| \leq j, |p_3-p_2| \leq p$, we may approximate the above

$$\begin{aligned} & (\frac{n^2}{f^4N})^2 \sum_{j_2,j_3,p_2,p_3} \frac{(2j_3+1)(2p_3+1)}{(j_3(j_3+1)+p_3(p_3+1))^2}(d^{(j)}_{j_3-j_2,0}(\theta_3))^2(d^{(p)}_{p_3-p_2,0}(\tilde{\theta}_3))^2 \\ = & (\frac{n^2}{f^4N})^2 \sum_{j_3,p_3} \frac{(2j_3+1)(2p_3+1)}{(j_3(j_3+1)+p_3(p_3+1))^2} \\ \sim & (\frac{n^2}{f^4N})^2 log(\frac{4l^2}{j^2+p^2}), \end{aligned} \tag{2.17}$$

where the lower cut-off is provided by the external momenta.

The non-planar amplitude can be estimated under the same approximation as

$$(\frac{n^2}{f^4N})^2 \sum_{j_2,j_3,p_2,p_3} e^{i\phi_{123}} \frac{(2j_2+1)(2p_2+1)}{(j_2(j_2+1)+p_2(p_2+1))(j_3(j_3+1)+p_3(p_3+1))} (d^{(j)}_{j_3-j_2,0}(\theta_3))^2(d^{(p)}_{p_3-p_2,0}(\tilde{\theta}_3))^2. \tag{2.18}$$

In an analogous way, it can be evaluated as

$$\begin{aligned} & (\frac{n^2}{f^4N})^2 \sum_{j_2,j_3,p_2,p_3} \frac{(2j_3+1)(2p_3+1)}{(j_3(j_3+1)+p_3(p_3+1))^2}e^{i\phi_{123}}(d^{(j)}_{j_3-j_2,0}(\theta_3))^2(d^{(p)}_{p_3-p_2,0}(\tilde{\theta}_3))^2 \\ = & (\frac{n^2}{f^4N})^2 \sum_{j_3,p_3} \frac{(2j_3+1)(2p_3+1)}{(j_3(j_3+1)+p_3(p_3+1))^2} d^{(j)}_{0,0}(\pi-2\theta_3)d^{(p)}_{0,0}(\pi-2\tilde{\theta}_3) \\ = & (\frac{n^2}{f^4N})^2 \sum_{j_3,p_3} \frac{(2j_3+1)(2p_3+1)}{(j_3(j_3+1)+p_3(p_3+1))^2} P_j\Big(1-\frac{j_3(j_3+1)}{2l(l+1)}\Big)P_p\Big(1-\frac{p_3(p_3+1)}{2l(l+1)}\Big). \end{aligned} \tag{2.19}$$

Since the Legendre polynomials $P_j(cos\theta)$ oscillate, we may identify the upper cut-off of the summations with the location of their first node. From the asymptotic behavior of the Legendre polynomials for large j:

$$P_j(cos\theta) \sim \sqrt{\frac{2}{j\pi sin\theta}} sin\Big((j+\frac{1}{2})\theta+\frac{\pi}{4}\Big), \tag{2.20}$$

we can estimate the location of their first node as $\theta \sim \pi/j$. We thus find the upper cut-off of the summations as $j_3 < l/j$ and $p_3 < l/p$. In this way, we obtain

$$(\frac{n^2}{f^4 N})^2 \int_{j^2}^{(\frac{l}{j})^2} dx \int_{p^2}^{(\frac{l}{p})^2} dy \frac{1}{(x+y)^2} \sim \frac{\lambda^4}{(4\pi)^4} log(\frac{l^2}{(j^2+p^2)^2}), \tag{2.21}$$

where $\lambda^2 = (4\pi)^2 n^2/f^4 N$ has been identified with the 't Hooft coupling [32].

The corresponding Wilson line of the NC gauge theory in the flat 4d space would be

$$Trexp(\frac{1}{l} ik \cdot A) Z^2. \tag{2.22}$$

Let us compute the corresponding two point function with the identical UV cut-off l. The planar amplitude would behave as

$$\frac{\lambda^4}{(2\pi)^4} \int^l d^4 q \frac{1}{q^2 (k+q)^2} \sim \frac{\lambda^4}{(4\pi)^2} log \frac{4l^2}{k^2}, \tag{2.23}$$

while that of the non-planar amplitude behaves as

$$\begin{aligned} & \frac{\lambda^4}{(2\pi)^4} \int^l d^4 q \frac{1}{q^2 (k+q)^2} exp(\frac{1}{l} iq \cdot k) \\ = & \frac{\lambda^4}{(4\pi)^2} \Big(\int_{k^2}^l dq^2 \frac{1}{q^2} + \int_l^{\frac{l^2}{k^2}} dq^2 \frac{1}{q^2} \Big) \\ = & \frac{\lambda^4}{(4\pi)^2} \Big(log(\frac{l}{k^2}) + log(\frac{l}{k^2}) \Big). \end{aligned} \tag{2.24}$$

We observe that the non-analytic behavior of the correlators (2.17) and (2.21) on $S^2 \times S^2$ and (2.23) and (2.24) on the flat 4d space are identical with the identification of the external momenta as $j^2 + p^2 \sim k^2$. This coincidence is expected to hold in the large N limit where $S^2 \times S^2$ becomes locally flat. The correlators on $S^2 \times S^2$ and the flat 4d space should agree as long as their momenta are large enough to probe a local region which is indistinguishable in the both cases.

In (2.24), we have separated the integral into small and large momentum contributions. The lower cut-off of the integral is provided by the external momenta. The small momentum contribution can be associated with the Wilson coefficient of the leading OPE expansion since the derivations with respect to the external momenta render it finite. On the other hand, the upper cut-off of the integral comes from the rapidly oscillating phase of the integrand. Due to the uncertainty relation in non-commutative space $\Delta x \Delta y \sim l$ and $\Delta x \Delta k \sim l$, a quantum which carries momentum larger than $\sqrt{l}$ extend the same amount in the perpendicular direction. Therefore the large momentum contribution can be interpreted in the dual

coordinate space where the upper cut-off l/k can be identified as a long distance cut-off. In this way we can identify it as the Fourier transformation of the long rang interaction $1/x^4$ as

$$\frac{\lambda^4}{(2\pi)^4}\int_{\sqrt{l}} d^4x \frac{1}{x^4} exp(\frac{1}{l} ik\cdot x). \tag{2.25}$$

We thus conclude that the two point function of the Wilson lines with $J = 2$ (2.8) behaves as follows for small external momenta to the leading order in perturbation theory

$$\frac{\lambda^4}{(4\pi)^4}\Big(2log(l/P^2) + log(l/P^2) + log(l)\Big). \tag{2.26}$$

The first term is identical to ordinary gauge theory if we identify the UV cut-off with non-commutative scale $\sqrt{l}$. The second term indicates a long range interaction due to UV/IR mixing specific to NC gauge theory as in (2.25). We discard the last term as it corresponds to δ function in coordinate space. This long range interaction is consistent with that of a Kaluza-Klein mode which couples to this operator in supergravity interpretation. The first term can be interpreted in terms of the Kaluza-Klein mode just like ordinary AdS/CFT correspondence. Nevertheless the second term represents the extra long range interaction which is absent in ordinary gauge theory. We investigate the supergravity description of NC gauge theory correlators in section 5.

The two point functions of generic chiral operators with $J > 2$ behave at tree level as

$$\begin{aligned} &\frac{n}{N} < Tr\mathcal{Y}_{j,p} z^J Tr\bar{z}^J \mathcal{Y}^\dagger_{j,p} > \\ = \ & < j,p| \prod_{i=1}^J \frac{1}{P_i^2} |j,p>_p + < j,p| \prod_{i=1}^J \frac{1}{P_i^2} |j,p>_{np}, \end{aligned} \tag{2.27}$$

where the planar amplitude is

$$\begin{aligned} < j,p| \prod_{i=1}^J \frac{1}{P_i^2} |j,p>_p \ &\equiv \ \frac{n^{J+1}}{f^{4J}N} \sum_{2\cdots J} \Phi^*_{12\cdots J}\Big(\prod_{i=1}^J \frac{1}{P_i^2}\Big)\Phi_{12\cdots J}, \\ \Phi_{12\cdots J} \ &\equiv \ Tr\mathcal{Y}_1\mathcal{Y}_2\cdots\mathcal{Y}_J, \end{aligned} \tag{2.28}$$

while the $J-1$ non-planar amplitudes are

$$\begin{aligned} &< j,p| \prod_{i=1}^J \frac{1}{P_i^2} |j,p>_{np} \\ \equiv \ & \frac{n^{J+1}}{f^{4J}N} \sum_{2\cdots J} \Big(\Phi^*_{23\cdots J1} + (J-2)\ cyclic\ permutations\Big)\Big(\prod_{i=1}^J \frac{1}{P_i^2}\Big)\Phi_{12\cdots J}. \end{aligned} \tag{2.29}$$

We can estimate non-analytic part of the planar amplitudes from the following recursive relation

$$
\begin{aligned}
&< j,p|\prod_{i=1}^{J}\frac{1}{P_i^2}|j,p>_p \\
= \ & \frac{n}{f^4}\sum_{j_1,p_1,j',p'}\left\{\begin{array}{ccc} j & j_1 & j' \\ l & l & l \end{array}\right\}^2\left\{\begin{array}{ccc} p & p_1 & p' \\ l & l & l \end{array}\right\}^2 \\
&\times\frac{(2j_1+1)(2p_1+1)(2j'+1)(2p'+1)}{j_1(j_1+1)+p_1(p_1+1)} < j',p'|\prod_{i=2}^{J}\frac{1}{P_i^2}|j',p'>_p . \qquad (2.30)
\end{aligned}
$$

Since

$$
\sum_{j_1}(2j_1+1)\left\{\begin{array}{ccc} j & j_1 & j' \\ l & l & l \end{array}\right\}^2 = \frac{1}{2l+1}, \qquad (2.31)
$$

and $|j_1 - j'| \le j$, we may evaluate it as

$$
\begin{aligned}
&< j,p|\prod_{i=1}^{J}\frac{1}{P_i^2}|j,p>_p \\
\sim \ & \frac{n^2}{f^4N}\sum_{j'^2+p'^2>P^2}\frac{(2j'+1)(2p'+1)}{j'(j'+1)+p'(p'+1)} < j',p'|\prod_{i=2}^{J}\frac{1}{P_i^2}|j',p'>_p \\
&+\frac{n^2}{f^4N}\sum_{j'^2+p'^2<P^2}\frac{(2j'+1)(2p'+1)}{P^2} < j',p'|\prod_{i=2}^{J}\frac{1}{P_i^2}|j',p'>_p . \qquad (2.32)
\end{aligned}
$$

From this recursion relation, we identify the non-analytic part as

$$
< j,p|\prod_{i=1}^{J}\frac{1}{P_i^2}|j,p>_p\sim (-1)^{J-2}\frac{\lambda^{2J}}{(4\pi)^{2J}}\frac{1}{(J-1)!(J-2)!}(P^2)^{J-2}log(\frac{4l^2}{P^2}), \qquad (2.33)
$$

corresponding to $1/x^{2J}$ behavior in real space.

As we have argued for the $J=2$ case, we should be able to reproduce these correlators from those on the flat 4d space in the large N limit. The planar amplitude can be estimated as

$$
\begin{aligned}
I_J \ &\equiv \ \lambda^{2J}\int\prod_{1}^{J}\frac{d^4k_i}{(2\pi)^4}\frac{1}{k_i^2}(2\pi)^4\delta^4(\sum_{i=1}^{J}k_i-k) \\
&= \ \lambda^{2J}\int\frac{dx^4}{l^4}exp(\frac{1}{l}ik\cdot x)\Big(\frac{l^2}{4\pi^2(x^2+\delta^2)}\Big)^J \\
&= \ \lambda^{2J}\frac{1}{\Gamma(J)}(\frac{1}{4\pi^2})^J l^{2(J-2)}\int_0^\infty d\alpha\alpha^{J-1}\int dx^4 exp(\frac{1}{l}ik\cdot x-\alpha(x^2+\delta^2)) \\
&= \ \lambda^{2J}\pi^2\frac{1}{\Gamma(J)}(\frac{1}{4\pi^2})^J l^{2(J-2)}\int_0^\infty d\alpha\alpha^{J-3}exp(-\frac{k^2}{4\alpha l^2}-\delta^2\alpha) \\
&= \ 2\lambda^{2J}\pi^2\frac{1}{\Gamma(J)}(\frac{1}{4\pi^2})^J(\frac{l^2}{\delta^2})^{J-2}(\frac{\delta^2k^2}{4l^2})^{\frac{J-2}{2}}K_{J-2}\Big(2\sqrt{\frac{\delta^2k^2}{4l^2}}\Big), \qquad (2.34)
\end{aligned}
$$

where we have introduced the short distance cut-off δ. The non-analytic part can be identified as

$$I_J \sim \frac{\lambda^{2J}}{(4\pi)^{2(J-1)}} \frac{1}{\Gamma(J)\Gamma(J-1)} (-k^2)^{J-2} log(\frac{4l^2}{k^2}), \tag{2.35}$$

where we have put the short distance cut-off $\delta \sim 1$. Alternatively we can obtain the identical result by performing the following partial integrations:

$$\begin{aligned}
& \frac{1}{16\pi^2}(-k^2)^{J-2} log(\frac{l^2}{k^2}) \\
= & \int \frac{d^4x}{l^4} (\frac{l^2}{4\pi^2 x^2})^2 (\partial^2)^{J-2} exp(\frac{1}{l} ik\cdot x) \\
= & \Gamma(J)\Gamma(J-1)(4\pi)^{2(J-2)} \int \frac{d^4x}{l^4} exp(\frac{1}{l} ik\cdot x)(\frac{l^2}{4\pi^2 x^2})^J. \qquad (2.36)
\end{aligned}$$

The non-planar amplitudes can be estimated by using the identical recursion relation with the planar amplitude case (2.32). However we need to use the different initial condition (2.21) which contains UV/IR mixing effect for the $J = 2$ case. We can indeed argue that the non-analytic behavior of this amplitude only comes from such a part of the phase space. In this way, we estimate each non-planar amplitude as

$$(-1)^{J-2} \frac{\lambda^{2J}}{(4\pi)^{2J}} \frac{1}{(J-1)!(J-2)!} (P^2)^{J-2} log(\frac{4l^2}{(P^2)^2}). \tag{2.37}$$

The non-planar contributions may also be estimated in the flat limit:

$$I_{n,m} = \lambda^{2J} \int \prod_{i=1}^{J} \frac{d^4k_i}{(2\pi)^4} \frac{1}{k_i^2} (2\pi)^4 \delta^4(\sum_{i=1}^{J} k_i - k) exp(\frac{1}{l} ik \wedge \sum_{i=1}^{m} k_i), \tag{2.38}$$

where $n + m = J$. It is convenient to estimate it in real space as

$$\begin{aligned}
& \lambda^{2J} \int \frac{dx^4}{l^4} exp(\frac{1}{l} ik\cdot x)\Big(\frac{l^2}{4\pi^2(x^2+\delta^2)}\Big)^n \Big(\frac{l^2}{4\pi^2((x-\tilde{\delta})^2+\delta^2)}\Big)^m \\
= & \lambda^{2J}\pi^2 \frac{1}{\Gamma(n)\Gamma(m)} (\frac{1}{4\pi^2})^J l^{2(J-2)} \int_0^\infty d\alpha\alpha^{n-1} d\beta\beta^{m-1} exp(-\frac{k^2}{4(\alpha+\beta)l^2} - \delta^2(\alpha+\beta) - \tilde{\delta}^2 \frac{\alpha\beta}{\alpha+\beta}) \\
= & \lambda^{2J}\pi^2 \frac{1}{\Gamma(n)\Gamma(m)} (\frac{1}{4\pi^2})^J l^{2(J-2)} \int_0^1 d\alpha\alpha^{n-1}(1-\alpha)^{m-1} \int_0^\infty d\lambda\lambda^{J-3} \\
& \times exp(-\frac{k^2}{4\lambda l^2} - \lambda\delta^2 - \lambda\alpha(1-\alpha)\tilde{\delta}^2), \qquad (2.39)
\end{aligned}$$

where $|\tilde{\delta}| = |k| > \delta$. The above expression is equal to

$$\lambda^{2J}\pi^2 \frac{1}{\Gamma(n)\Gamma(m)} (\frac{1}{4\pi^2})^J \int_0^1 d\alpha\alpha^{n-1}(1-\alpha)^{m-1} 2(\frac{l^2}{\bar{\delta}^2})^{J-2} (\frac{\bar{\delta}^2 k^2}{4l^2})^{\frac{J-2}{2}} K_{J-2}\Big(2\sqrt{\frac{\bar{\delta}^2 k^2}{4l^2}}\Big), \tag{2.40}$$

where $\bar{\delta}^2 = \delta^2 + \alpha(1-\alpha)\tilde{\delta}^2$. The singular part of the modified Bessel function behaves as

$$\begin{aligned} & K_{J-2}\Big(2\sqrt{\frac{\bar{\delta}^2 k^2}{4l^2}}\Big) \\ = \;& \frac{(-1)^{J-1}}{2(J-2)!}\Big(\frac{\bar{\delta}^2 k^2}{4l^2}\Big)^{\frac{J-2}{2}} log(\frac{\bar{\delta}^2 k^2}{4l^2}) + \frac{(J-3)!}{2}\Big(\frac{\bar{\delta}^2 k^2}{4l^2}\Big)^{-\frac{(J-2)}{2}} + \cdots, \end{aligned} \tag{2.41}$$

where $\cdots$ denotes higher order terms in k^2. Since the second term in the last expression gives rise to the short distance singularity as

$$\lambda^{2J}\pi^2(\frac{1}{4\pi^2})^J \frac{1}{(J-1)(J-2)}(\frac{l^2}{\delta^2})^{J-2} + \cdots, \tag{2.42}$$

the non-analytic behavior of the amplitude comes from the first term as

$$\frac{\lambda^{2J}}{(4\pi)^{2(J-1)}}\frac{1}{\Gamma(J)\Gamma(J-1)}(-k^2)^{J-2} log(\frac{l^2}{(k^2)^2}). \tag{2.43}$$

We note that we could contemplate more generic extension of the chiral operators in NC gauge theory on the flat 4d space such as

$$Tr exp(\frac{1}{l}i\alpha_1 k\cdot A)Z exp(\frac{1}{l}i\alpha_2 k\cdot A)Z\cdots exp(\frac{1}{l}i\alpha_J k\cdot A)Z, \tag{2.44}$$

where $\sum\alpha_i = 1$. This operator would correspond to $Tr\mathcal{Y}_{j_1 j_1} z \mathcal{Y}_{j_2 j_2} z \cdots \mathcal{Y}_{j_J j_J} z$ on $S^2\times S^2$ where $\sum j_i = j$. For a generic choice of α_i, we can observe by power counting arguments that the non-planar contributions to the two point correlators behave when $\delta\to 0$ as

$$\lambda^{2J}\int\frac{dx^4}{l^4} exp(\frac{1}{l}ik\cdot x)\prod_{i=1}^{J}\frac{l^2}{4\pi^2((x-\tilde{\delta}_i)^2+\delta^2)} \sim \lambda^{2J}(\frac{l^2}{k^2})^{J-2}, \tag{2.45}$$

where $|\tilde{\delta}_i| = \alpha_i|k|$. Since the planar contributions are identical to those in ordinary gauge theory, such contributions completely alter the non-analytic behavior of the two point functions. We therefore restrict our considerations to $Tr\mathcal{Y}z^J$ type operators in this paper.

By putting together planar and non-planar contributions: (2.33) and (2.37), we obtain

$$\begin{aligned} & \frac{n}{N} < Tr\mathcal{Y}_{j,p} z^J Tr\bar{z}^J \mathcal{Y}^\dagger_{j,p} > \\ \sim \;& (-P^2)^{J-2} log(\frac{l^2}{P^2}) + (J-1)(-P^2)^{J-2} log(\frac{l^2}{(P^2)^2}). \end{aligned} \tag{2.46}$$

We can rewrite it as follows just like $J=2$ case in (2.26)

$$J(-P^2)^{J-2} log(\frac{l}{P^2}) + (J-1)(-P^2)^{J-2} log(\frac{l}{P^2}) + (-P^2)^{J-2} log(l). \tag{2.47}$$

We find that the first term is identical to ordinary gauge theory with UV cut-off of $\sqrt{l}$. The second term is specific to NC gauge theory representing the long range interaction due to UV/IR mixing. It is consistent to interpret it in terms of a Kaluza-Klein mode which couples to this operator. The third term does not correspond to long distance physics as it corresponds to the derivatives of δ function.

3 Quantum corrections to the correlators

In this section, we investigate the quantum corrections to the two point correlators of the Wilson lines at the one loop level. For this purpose, we need to consider the renormalization of NC gauge theory on $S^2 \times S^2$. At the one loop level, we have found the divergence of the gauge field self-energy type in the planar sector [35] [2]. In order to remove it, we may renormalize the gauge field as

$$\begin{aligned} A_\mu &= f(p_\mu + Z_{\Delta^2} a_\mu^{j,p} \mathcal{Y}_{j,p}) \\ Z_{\Delta^2} &= 1 - \frac{1}{8\pi^2}\lambda^2 log\Big(\frac{N}{n\Delta^2}\Big), \end{aligned} \tag{3.1}$$

where Δ is a renormalization scale.

This renormalization procedure does not remove the divergence of the non-planar self-energy of the $SU(n)$ singlet field. Such a divergence cannot be removed by a constant wave function renormalization of the $SU(n)$ singlet field since it occurs only when $p^2 < l$. This phenomenon is a manifestation of the UV/IR mixing in NC gauge theory. In other words, the gauge symmetry of NC gauge theory reduces to ordinary gauge symmetry in the small momentum limit. In such a limit, $U(n)$ gauge symmetry decouples as $SU(n) \times U(1)$. It is therefore expected that the singlet field is renormalized differently from the non-singlet fields in the small momentum regime.

We choose to let the divergent self-energy of the singlet field in the small momentum regime as it is. We believe that this procedure does not spoil the renormalizability of the theory since the singlet field decouples from the non-singlet fields in the low momentum regime. As a concrete example, we consider the two point functions with $J = 1$ to which the singlet field contributes. With generic momenta, they behave at the tree level as

$$\frac{n}{N} < Tr\mathcal{Y}_{j,p} z Tr\bar{z}\mathcal{Y}^\dagger_{j,p} >= \frac{n}{f^4 N}\frac{1}{P^2}, \tag{3.2}$$

[2] We ignore mass and Chern-Simons type terms by focusing on large enough momentum scale.

where $P^2 = j(j+1) + p(p+1)$. At the one loop level, we have found the following quantum corrections to this correlator in our renormalization procedure when $P^2 << l$

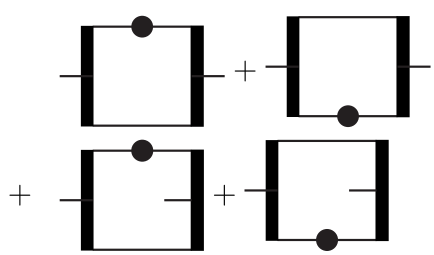
$$\begin{aligned} & \frac{n}{f^4 N}\Big(Z^2_{P^2} - \omega_{np}(P^2)\Big)\frac{1}{P^2} \\ = \; & \frac{1}{n}\frac{\lambda^2}{(4\pi)^2}\Big(1 - \frac{\lambda^2}{4\pi^2}log(P^2)\Big)\frac{1}{P^2}, \end{aligned} \tag{3.3}$$

where we have taken the renormalization scale $\Delta^2 = P^2$ and $\omega_{np}(P^2)$ is the non-planar contribution to gauge field self-energy. The planar self-energy has been cancelled by the counter term with this renormalization procedure. We find that the correlator (3.3) requires no infinite renormalization since it is independent of N with fixed 't Hooft coupling λ^2. However we find a non-analytic finite correction. Since we are dealing with non-local operators, there is always a question as which operator corresponds to local operators. We can indeed eliminate $\Big(1 - \lambda^2 log(P^2)/4\pi^2\Big)$ factor by redefining the $J = 1$ operator.

$$Tr\mathcal{Y}z \to \Big(1 - \frac{\lambda^2}{8\pi^2}log(P^2)\Big)Tr\mathcal{Y}z. \tag{3.4}$$

In fact such a freedom exists for the Wilson lines since the operators with different momenta are independent of each other. In perturbation theory, we have found it possible to rescale the operators with $J = 1$ in such a way that they can be interpreted as the Fourier transformation of a local operator with no anomalous dimensions. We remark that there is no $J = 1$ chiral operator in ordinary gauge theory with $SU(n)$ gauge group to which AdS/CFT correspondence applies. The supergravity description of this operator has been studied in [36].

The most relevant chiral operator in ordinary gauge theory starts with $J = 2$. We move on to investigate the quantum corrections to the two point correlators of the noncommutative analogue of the $J = 2$ chiral operator. The one loop self-energy corrections to the propagators are

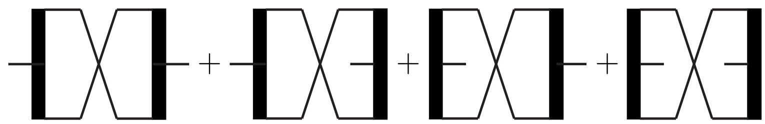


$$\begin{aligned} = \; & -8\frac{n^4}{f^{12}N}\sum_{2345}\Psi^*_{123}\Psi^*_{\bar{2}45}\frac{1}{P_2^2P_3^2P_4^2P_5^2}\Psi_{\bar{2}45}\Psi_{123} \\ & -8\frac{n^4}{f^{12}N}\sum_{2345}e^{i\phi_{123}}\Psi^*_{123}\Psi^*_{\bar{2}45}\frac{1}{P_2^2P_3^2P_4^2P_5^2}\Psi_{\bar{2}45}\Psi_{123}, \end{aligned} \tag{3.5}$$

where the operators P_i do not act on the matrices labeled by $\bar{i}$ in our convention. The other quantum corrections arise due to the following interaction terms of the action.

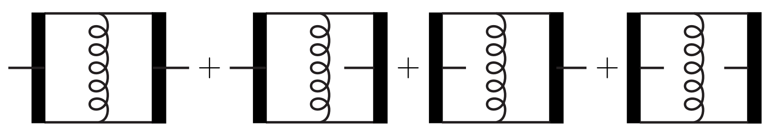
$$Tr\frac{1}{2}[\bar{Z}, Z]^2 - [A_\alpha, \bar{Z}][A^\alpha, Z]. \tag{3.6}$$

The quartic vertex gives rise to



$$\begin{aligned}
= \quad & \frac{n^4}{f^{12}N}\sum_{2345}\frac{1}{P_2^2P_3^2P_4^2P_5^2}\Psi_{45\bar{3}\bar{2}} \\
& \times\Big(\Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{123} + \Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{132} + \Psi_{\bar{1}\bar{4}\bar{5}}\Psi_{123} + \Psi_{\bar{1}\bar{4}\bar{5}}\Psi_{132}\Big).
\end{aligned} \tag{3.7}$$

The cubic vertices give



$$\begin{aligned}
= \quad & -\frac{n^4}{f^{12}N}\sum_{23456}\frac{(P_4+P_2)\cdot(P_3+P_5)}{P_2^2P_3^2P_4^2P_5^2P_6^2}\Psi_{\bar{2}46}\Psi_{5\bar{3}\bar{6}} \\
& \times\Big(\Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{123} + \Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{132} + \Psi_{\bar{1}\bar{4}\bar{5}}\Psi_{123} + \Psi_{\bar{1}\bar{4}\bar{5}}\Psi_{132}\Big).
\end{aligned} \tag{3.8}$$

By combing the above two, the leading contribution is found as

$$\begin{aligned}
& \frac{n^4}{f^{12}N}\sum_{23456}\frac{P_2+P_3^2+P_4^2+P_5^2}{P_2^2P_3^2P_4^2P_5^2P_6^2}\Psi_{\bar{2}46}\Psi_{5\bar{3}\bar{6}} \\
& \times\Big(\Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{123} + \Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{132} + \Psi_{\bar{1}\bar{4}\bar{5}}\Psi_{123} + \Psi_{\bar{1}\bar{4}\bar{5}}\Psi_{132}\Big) \\
= \quad & 4\frac{n^4}{f^{12}N}\sum_{2346}\Psi^*_{123}\Psi^*_{\bar{2}46}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{123} \\
& +4\frac{n^4}{f^{12}N}\sum_{2346}\Psi^*_{123}\Psi^*_{\bar{2}46}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{123}e^{i\phi_{123}} \\
& +4\frac{n^4}{f^{12}N}\sum_{23456}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{5\bar{3}\bar{6}}\Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{123}e^{i\phi_{145}} \\
& +4\frac{n^4}{f^{12}N}\sum_{23456}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{5\bar{3}\bar{6}}\Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{123}e^{i\phi_{123}}e^{i\phi_{145}},
\end{aligned} \tag{3.9}$$

where $e^{i\phi_{123}} = (-1)^{j_1+j_2+j_3+p_1+p_2+p_3}$ and $e^{i\phi_{145}} = (-1)^{j_1+j_4+j_5+p_1+p_4+p_5}$. Since the logarithmically divergent contribution in the first and second lines of the right-hand side in (3.9) cancels only the half of (3.5), we have found that the two point function of the $J = 2$ chiral

operators is renormalized at the one loop level as

$$
\begin{aligned}
&-4\frac{n^4}{f^{12}N}\sum_{2346}\Psi^*_{123}\Psi^*_{\bar{2}46}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{123}\\
&-4\frac{n^4}{f^{12}N}\sum_{2346}\Psi^*_{123}\Psi^*_{\bar{2}46}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{123}e^{i\phi_{123}}\\
&+4\frac{n^4}{f^{12}N}\sum_{23456}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{5\bar{3}\bar{6}}\Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{123}e^{i\phi_{145}}\\
&+4\frac{n^4}{f^{12}N}\sum_{23456}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{5\bar{3}\bar{6}}\Psi_{\bar{1}\bar{5}\bar{4}}\Psi_{123}e^{i\phi_{123}}e^{i\phi_{145}}.
\end{aligned}
\tag{3.10}
$$

We are considering the case when the external momenta $P^2 << l$. In such a situation, we may adopt Edmonds' approximation to compare $\Psi_{\bar{1}\bar{5}\bar{4}}$ and $\Psi_{\bar{1}\bar{4}\bar{5}}$. After extracting the phase $e^{i\phi_{145}}$ from the $3j$ symbols, the former contains a factor

$$\left\{ \begin{array}{ccc} j & j_4 & j_5 \\ l & l & l \end{array} \right\} \sim \frac{(-1)^{j_5}}{\sqrt{(2j_5+1)(2l+1)}} d^{(j)}_{j_5-j_4,0}(\theta_5), \tag{3.11}$$

while that of the latter is

$$\left\{ \begin{array}{ccc} j & j_4 & j_5 \\ l & l & l \end{array} \right\} (-1)^{j+j_4+j_5} \sim \frac{(-1)^{j_5}}{\sqrt{(2j_5+1)(2l+1)}} d^{(j)}_{j_5-j_4,0}(\pi-\theta_5). \tag{3.12}$$

The rotation matrices are given by the associated Legendre functions

$$d^{(j)}_{m,0}(\theta) = \Big[\frac{(j-m)!}{(j+m)!}\Big]^{\frac{1}{2}} P^m_j(cos(\theta)). \tag{3.13}$$

We would like to find out the condition when these two expressions agree. For this purpose, we make use of the asymptotic behavior of the associated Legendre functions for large j:

$$P^m_j(cos(\theta)) \sim (-j)^m \sqrt{\frac{2}{j\pi sin\theta}} cos\Big((j+\frac{1}{2})\theta + \frac{m\pi}{2} - \frac{\pi}{4}\Big). \tag{3.14}$$

From this expression, we can infer that the distance of the neighboring nodes is π/j. We argue that the planar and the non-planar amplitudes are coherent when the difference of the arguments of the associated Legendre functions (θ_5 and $\pi-\theta_5$) is smaller than it. Such a requirement leads to the condition $j_4, j_5 < \pi l/j$ and $p_4, p_5 < \pi l/p$ as well. This argument agrees with the estimate based on the effect of non-commutative phase $exp(ik \wedge p/l)$ in the flat limit.

From these arguments, we can estimate (3.10) as

$$
\begin{aligned}
&-4\frac{n^4}{f^{12}N}\sum_{234'6}\Psi^*_{123}\Psi^*_{\bar{2}46}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{123}\\
&-4\frac{n^4}{f^{12}N}\sum_{234'6}\Psi^*_{123}\Psi^*_{\bar{2}46}\frac{1}{P_2^2P_3^2P_4^2P_6^2}\Psi_{\bar{2}46}\Psi_{123}e^{i\phi_{123}},
\end{aligned}
\tag{3.15}
$$

where $\sum_{234'6}$ implies that the summation is constrained with respect to j_4, p_4 as $l/j < j_4 < l, l/p < p_4 < l$. With such a restriction, we can evaluate the following expression as

$$\frac{4n}{f^4}\sum_{4'6}\Psi^*_{\bar{2}46}\frac{1}{P_4^2P_6^2}\Psi_{\bar{2}46} \sim \frac{\lambda^2}{4\pi^2}log(P^2). \tag{3.16}$$

From these considerations, we conclude that the one loop correction to the two point function of the $J = 2$ chiral operators is

$$-\frac{\lambda^2}{4\pi^2}log(P^2) \times tree\ result\ (2.26). \tag{3.17}$$

Since it is proportional to a finite factor $log(P^2)$, this operator requires no infinite renormalization at the one loop level. Nevertheless we find non-analytic finite corrections. We may eliminate $\left(1 - \lambda^2 log(P^2)/4\pi^2\right)$ factor by redefining the $J = 2$ operator just like $J = 1$ case:

$$Tr\mathcal{Y}z^2 \to \left(1 - \frac{\lambda^2}{8\pi^2}log(P^2)\right)Tr\mathcal{Y}z^2. \tag{3.18}$$

In perturbation theory, we have found it possible again to rescale the operators with $J = 2$ in such a way that they can be interpreted as the Fourier transformation of a local operator with no anomalous dimensions.

We can further argue that the leading one loop corrections to the correlators with $J > 2$ are just analogous to the $J = 2$ case. The one loop corrections in the planar sector with respect to gauge indices are local in the string world sheet sense as they involve two neighboring Z fields. The chiral operators can be decomposed into J sections each of which is bounded by two Z fields. Since each Z field is shared by two sections, the quarter of the self-energy correction to each Z field can be associated with a section. In the commutative limit, such self-energy corrections are cancelled by the rest of the one loop corrections in each section. Therefore we only need to consider the one loop corrections which are affected by the non-commutative phase. Furthermore it is sufficient to consider the renormalization of each operator.

We can thus focus on a section of an operator which contains the spherical harmonics $\mathcal{Y}$. As the $J = 2$ case, the one loop corrections can be divided into the self-energy corrections and the rest. Were it not for $\mathcal{Y}$, they cancel each other. With the presence of $\mathcal{Y}$, we need to interchange $\mathcal{Y}$ and Z field in this process. After such an operation, the non-commutative phase could arise which cuts-off the momentum integration. Let us expand Z field by matrices $\mathcal{Y}_1$. We thus need to estimate the phase difference between $\mathcal{Y}\mathcal{Y}_1$ and $\mathcal{Y}_1\mathcal{Y}$. For this

purpose, we may compare $Tr\mathcal{Y}\mathcal{Y}_1\mathcal{Y}_2$ and $Tr\mathcal{Y}_1\mathcal{Y}\mathcal{Y}_2$. By the identical argument which was used to estimate the phase difference of $\Psi_{\bar{1}5\bar{4}}$ and $\Psi_{\bar{1}\bar{4}\bar{5}}$, we estimate that they are coherent when $j_1, j_2 < \pi l/j$ and $p_1, p_2 < \pi l/p$. Since the upper momentum integration cut-off has been lowered in this way, the renormalization factor differs by the term $-\frac{\lambda^2}{8\pi^2}log(P^2)$ in comparison to the other sections. The two point functions of the generic chiral operators with $J \geq 2$ are thus renormalized as follows:

$$-\frac{\lambda^2}{4\pi^2}log(P^2) \times tree\ result\ (2.47). \tag{3.19}$$

We may eliminate the non-analytic finite factor $\left(1 - \lambda^2 log(P^2)/4\pi^2\right)$ by redefining the $J \geq 2$ operators in general as:

$$Tr\mathcal{Y}z^J \rightarrow \left(1 - \frac{\lambda^2}{8\pi^2}log(P^2)\right)Tr\mathcal{Y}z^J. \tag{3.20}$$

We conclude that we have found it possible to rescale the operators with $J \geq 2$ in such a way that they can be interpreted as the Fourier transformation of the local operators with no anomalous dimensions.

Although we can rescale the operators by momentum dependent factors to fit any two point functions as we wish, such a procedure alters multi-point functions. We argue that our procedure is legitimate since it removes one loop quantum corrections of all correlators since the renormalization effect is associated with individual operators. It is certainly necessary to make contact with supergravity since supergravity predicts vanishing anomalous dimensions in the small momentum regime.

We can also study the following operators on $S^2 \times S^2$.

$$y_{jm}^{\alpha_1,\alpha_2,\cdots,\alpha_j} y_{pq}^{\beta_1,\beta_2,\cdots,\beta_p} TrA_{\alpha_1}A_{\alpha_2}\cdots A_{\alpha_j}A_{\beta_1}A_{\beta_2}\cdots A_{\beta_p}\Phi_{i_1}\Phi_{i_2}\cdots, \tag{3.21}$$

where $\Phi_1 = (A_6 + iA_7)/\sqrt{2}$ and $\Phi_2 = Z = (A_8 + iA_9)/\sqrt{2}$. We expand $A_\mu = f(p+a)_\mu$ as before and the leading terms are

$$Tr\mathcal{Y}_{jp}\phi_{i_1}\phi_{i_2}\cdots, \tag{3.22}$$

where we have rescaled the operators to remove extra factors of f. In ordinary gauge theory, these operators can be identified with the states of the spin chain by associating ϕ_1 with up spins and ϕ_2 with down spins.

Since the one loop planar corrections with respect to gauge indices are local in the sense that they involve adjacent ϕ fields, it is sufficient to focus on a section of the operator.

We first consider the renormalization of a section which contains no $\mathcal{Y}$. Apart from the self-energy corrections of ϕ fields, we need to compute the quantum corrections due to the following interaction terms of the action

$$Tr\Big(\frac{1}{2}[{\Phi_1}^\dagger, \Phi_1]^2 + \frac{1}{2}[{\Phi_2}^\dagger, \Phi_2]^2 - [A_\alpha, {\Phi_i}^\dagger][A^\alpha, \Phi_i]$$
$$-[\Phi_1^\dagger, \Phi_2^\dagger][\Phi_1, \Phi_2] - [\Phi_1^\dagger, \Phi_2][\Phi_1, \Phi_2^\dagger]\Big). \tag{3.23}$$

The quartic vertices give rise to

$$-\frac{n^3}{f^{12}N}\sum_{45}\frac{1}{P_4^2P_5^2}\Psi_{145}\Psi_{\bar{5}\bar{4}23}(1-T)$$
$$+\frac{n^3}{f^{12}N}\sum_{45}\frac{1}{P_4^2P_5^2}\Psi_{145}\Psi_{\bar{5}\bar{4}23}T, \tag{3.24}$$

where T is the exchange operator. The cubic vertices give

$$-\frac{n^3}{f^{12}N}\sum_{456}\frac{(P_4+P_2)\cdot(P_3+P_5)}{P_4^2P_5^2P_6^2}\Psi_{145}\Psi_{26\bar{4}}\Psi_{3\bar{5}\bar{6}}. \tag{3.25}$$

After including the quarter of the self-energy corrections of ϕ fields which are associated with this section, we find the following quantum correction:

$$-\frac{2n^3}{f^{12}N}\sum_{45}\frac{1}{P_4^2P_5^2}\Psi_{145}\Psi_{\bar{5}\bar{4}23}(1-T)$$
$$= \quad -\frac{n^2}{2f^8N}\omega(P^2)\Psi_{123}(1-T). \tag{3.26}$$

This is identical to the ordinary gauge theory result giving rise to the anomalous dimension of the spin chain Hamiltonian type [6][7][11]:

$$H = J + \frac{\lambda^2}{8\pi^2}\sum_i(1-T_i) = J + \frac{\lambda^2}{16\pi^2}\sum_i(1-\sigma_i\cdot\sigma_{i+1}). \tag{3.27}$$

In NC gauge theory, we need to study the renormalization effect of the section which contains $\mathcal{Y}$ as well. Just like the non-commutative chiral operator case, it gives rise to an extra renormalization factor

$$-\frac{\lambda^2}{8\pi^2}log(P^2) + \frac{\lambda^2}{8\pi^2}log(P^2)(1-T_1), \tag{3.28}$$

where T_1 exchanges the two fields adjacent to $\mathcal{Y}$. We can remove this factor by redefining the Wilson line operator W

$$W \to \Big(1 - \frac{\lambda^2}{8\pi^2}log(P^2) + \frac{\lambda^2}{8\pi^2}log(P^2)(1-T_1)\Big)W. \tag{3.29}$$

In this case also we can rescale the operators in such a way that they can be interpreted as the Fourier transformation of the local operators whose anomalous dimensions are given by the spin chain Hamiltonian of ordinary gauge theory.

So far we have neglected the gauge fields a_μ in the expansion of A_μ around the classical solution p_μ as $A = f(a+p)_\mu$. Before concluding this section, we investigate the effect of the gauge fields to the Wilson line correlators. The leading correction to (2.5) is

$$\begin{aligned} & y_{j,j} y_{p,p} Tr\Big(\sum_k (p_+)^k a_+ (p_+)^{j-k-1} (\tilde{p}_+)^p + \sum_k (p_+)^j (\tilde{p}_+)^k \tilde{a}_+ (\tilde{p}_+)^{p-k-1}\Big) z^J \\ \sim \ & \frac{1}{l} Tr\Big(\sum_k \mathcal{Y}_{k,0} a_+ \mathcal{Y}_{j-k-1,p} + \sum_k \mathcal{Y}_{j,k} a_+ \mathcal{Y}_{0,p-k-1}\Big) z^J. \end{aligned} \tag{3.30}$$

From this structure, we can estimate the magnitude of the gauge field effects to the two point functions as

$$\lambda^2 \frac{(P^2)^2}{l^2}. \tag{3.31}$$

We can conclude that these corrections can be neglected as long as the above quantity is small. It implies that the gauge field effects can be neglected when the operator probes the distance scale larger than R where $R^4 = \lambda^2 l^2$. We remark that R coincides with the radius of the background in dual supergravity which will be studied in the next section.

4 Supergravity description

It is an attractive idea that non-commutative gauge theories may also possess dual supergravity descriptions [20][21][30][41]. We have indeed demonstrated that we can perturbatively identify the non-commutative extensions of the BPS operators with no anomalous dimensions in this paper. These observables with finite momenta probe the low energy limit of NC gauge theory since the non-commutative scale is $O(N^{\frac{1}{4}})$. In NC gauge theory, there are both IR and UV/IR contributions to the two point correlators as we have seen in the preceding sections. Since the IR contributions are identical to those in ordinary gauge theory, the supergravity background need not change from AdS_5 were it not for UV/IR contributions. We will argue in what follows that the way it deviates from AdS_5 is consistent to accommodate UV/IR contributions.

We recall the Euclidean IIB supergravity action:

$$S_{IIB} \ = \ S_{NS} + S_R + S_{CS},$$

$$
\begin{aligned}
S_{NS} &= -\frac{1}{2}\int d^{10}x\sqrt{g}e^{-2\phi}(R+4\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}{H_3}^2),\\
S_R &= \frac{1}{4}\int d^{10}x\sqrt{g}({F_1}^2+\tilde{F}_3^2+\frac{1}{2}\tilde{F}_5^2),\\
S_{CS} &= \frac{1}{4}\int C_4\wedge H_3\wedge F_3,
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
\tilde{F}_3 &= F_3 - C_0\wedge H_3,\\
\tilde{F}_5 &= F_5-\frac{1}{2}C_2\wedge H_3+\frac{1}{2}B_2\wedge F_3.
\end{aligned}
\tag{4.2}
$$

Supergravity solution which is dual to $U(n)$ $NCYM_4$ is

$$
\begin{aligned}
e^\phi &= (\frac{ng^2}{r^4})\frac{1}{(1+\frac{ng}{r^4})},\\
\frac{1}{\alpha'}ds^2 &= (\frac{ng}{r^4})^{\frac{1}{2}}(\frac{d\vec{x}^2}{1+\frac{ng}{r^4}}+dr^2+r^2d\Omega_5^2),\\
B_2 &= \frac{1}{(1+\frac{ng}{r^4})}dx\wedge dy+\frac{1}{(1+\frac{ng}{r^4})}dz\wedge d\tau,\\
C_2 &= i\frac{1}{g}B_2,\\
C_0 &= -i\frac{r^4}{ng^2},\\
F_{0123r} &= -4i\frac{1}{(1+\frac{ng}{r^4})^2}\frac{n}{r^5}.
\end{aligned}
\tag{4.3}
$$

Here we have put NS B field strength $b=1$ which implies that the noncommutativity scale l_{NC} is $O(1)$. Since $S^2\times S^2$ approaches the flat 4d space in the large N limit, we believe that this solution is relevant to our problem.

Since ng which corresponds to the 't Hooft coupling of $NCYM_4$ sets the radius of 'AdS_5' and S_5 as $R^4/l_{NC}^4 = ng$, supergravity description is expected to be valid in the strong coupling limit. It is because the mass scale for the Kaluza-Klein modes can be estimated to be of order $1/R$ in comparison to that of the oscillator modes. In NC gauge theory, we have observed in the preceding section that the gauge field effects to the Wilson line correlators are small if they probe the distance scale larger than R where $R^4=\lambda^2 l^2$. Hence the identification of ng and λ^2 is indeed consistent since the non-commutative scale is $l_{NC}\sim\sqrt{l}$ in NC gauge theory. The supergravity regime corresponds to local field theory regime in NC gauge theory. We may further assume that n is large in order to keep the dilaton expectation value to be small.

It is useful to introduce the coordinate system where the five dimensional subspace $(\vec{x}, \rho)$ is conformally flat

$$\frac{1}{\alpha'}ds^2 = A(\rho)(d\vec{x}^2 + d\rho^2) + R^2 d\Omega_5^2. \tag{4.4}$$

Since

$$\rho = \int_R^r dr\sqrt{1 + \frac{R^4}{r^4}}, \tag{4.5}$$

we find that

$$A(\rho) \sim R^2/\rho^2, \quad \rho \to \pm\infty. \tag{4.6}$$

$A(\rho)$ has the unique maximum at $\rho = 0$ $(r = R)$.

For simplicity, we consider a massless field φ such as dilaton or graviton. It is reasonable to assume that they provide us generic information for the entire supergravity multiplets. Such a field φ obeys the following equation of motion

$$\frac{1}{2}\nabla^\mu\nabla_\mu\varphi - \nabla^\mu\phi\nabla_\mu\varphi = 0. \tag{4.7}$$

Eq.(4.7) can be rewritten as:

$$\begin{aligned} H\varphi &= 0, \\ H &= -\frac{A}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu + 2Ag^{\mu\nu}\partial_\mu\phi\partial_\nu. \end{aligned} \tag{4.8}$$

The Hamiltonian is

$$H = -(\vec{\nabla}^2 + \frac{\partial^2}{\partial\rho^2} + (\frac{3}{2}\frac{A'}{A} - 2\phi')\frac{\partial}{\partial\rho} + \frac{A}{R^2}\hat{L}^2), \tag{4.9}$$

where $A' = \partial A/\partial\rho$ and $\phi' = \partial\phi/\partial\rho$. The symbols $\vec{\nabla}^2$ and $\hat{L}^2$ denote the Laplacians on the flat 4d space and S^5 respectively.

We concentrate on the S wave of S^5 in what follows. The eigenfunction of H is found as $exp(ik\cdot x)\varphi$ with the eigenvalue $k^2 + E$. The eigenvalue E and its eigenfunction can be determined by solving the following quantum mechanics problem.

$$-(\frac{\partial^2}{\partial\rho^2} + (\frac{3}{2}\frac{A'}{A} - 2\phi')\frac{\partial}{\partial\rho})\varphi = E\varphi. \tag{4.10}$$

We first investigate the behavior of the solutions in the asymptotic regions $\rho \sim \pm\infty$. When $\rho \sim -\infty$, the Hamiltonian is well approximated as

$$(-\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho})\varphi = E\varphi. \tag{4.11}$$

The solutions are given by the Bessel functions $\rho^2\sqrt{\omega}J_2(\omega\rho)$ and $\rho^2\sqrt{\omega}Y_2(\omega\rho)$ where $\omega^2 = E$. They behave as $\varphi \sim |\rho|^{\frac{3}{2}}e^{\pm i\omega\rho}$ for large $|\rho|$. In the asymptotic region $\rho \sim \infty$, the Hamiltonian becomes

$$(-\frac{\partial^2}{\partial\rho^2} - \frac{5}{\rho}\frac{\partial}{\partial\rho})\varphi = E\varphi. \tag{4.12}$$

The solutions are again given by the Bessel functions $\rho^{-2}\sqrt{\omega}J_2(\omega\rho)$ and $\rho^{-2}\sqrt{\omega}Y_2(\omega\rho)$. Such wave functions behave as $\varphi \sim \rho^{-\frac{5}{2}}e^{\pm i\omega\rho}$ for $\rho \sim \infty$.

We may construct the propagator as follows

$$G(x, y) = \sum_j < x|j > \frac{1}{E_j} < j|y >, \tag{4.13}$$

where $|j>$ is the eigenstate of H with the eigenvalue E_j. With $< \rho|j >= \rho^2\sqrt{\omega}J_2(\omega\rho)$, the bulk propagator for negative ρ (or small r) which appears in ordinary AdS/CFT correspondence is obtained:

$$\begin{aligned} G(x, y)_B &= \frac{1}{R^8}\int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y}))\int_0^\infty d\omega\ \omega\frac{1}{\vec{k}^2+\omega^2} \\ &\quad \times\rho^2 J_2(\omega\rho){\rho'}^2 J_2(\omega\rho'). \end{aligned} \tag{4.14}$$

In fact it satisfies the desired equation in AdS_5

$$-\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu G(x, y)_B = \frac{1}{\sqrt{g}}\delta(x-y), \tag{4.15}$$

since the Bessel functions satisfy the following completeness condition:

$$\delta(\rho-\rho') = \int_0^\infty d\omega\ \omega\sqrt{\rho}J_2(\omega\rho)\sqrt{\rho'}J_2(\omega\rho'). \tag{4.16}$$

We may reexpress the propagator as follows

$$\begin{aligned} G(x, y)_B &= \frac{1}{2R^8}\int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y}))\int_{-\infty}^\infty d\omega\ \omega\frac{1}{\vec{k}^2+\omega^2} \\ &\quad \times\rho_<^2 J_2(\omega\rho_<)\rho_>^2 H_2^{(1)}(\omega\rho_>), \end{aligned} \tag{4.17}$$

where $\rho_<(\rho_>)$ denotes the smaller (larger) quantity between $|\rho|$ and $|\rho'|$.

In this form, we can now pick the residue of the simple pole at $\omega = ik$ to estimate its long distance behavior

$$\begin{aligned} G(x, y)_B &= \frac{1}{R^8}\int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y}))\rho_<^2 I_2(k\rho_<)\rho_>^2 K_2(k\rho_>) \\ &\sim \frac{3}{2\pi^2R^8}\frac{\rho^4\rho'^4}{(\vec{x}-\vec{y})^8}. \end{aligned} \tag{4.18}$$

Alternatively it can be directly estimated from (4.14) as

$$
\begin{aligned}
& \int d\omega\ \omega \frac{1}{\vec{k}^2+\omega^2}\rho^2 J_2(\omega\rho){\rho'}^2 J_2(\omega\rho') \\
\sim\ & \frac{\rho^4\rho'^4}{64}\int d\omega\ \omega^5 \frac{1}{\vec{k}^2+\omega^2} \\
\sim\ & -\frac{\rho^4\rho'^4}{128}(k^2)^2 logk^2, \qquad (4.19)
\end{aligned}
$$

where we have retained the non-analytic part in k^2 which reproduces the long range interaction (4.18) after the Fourier transformation.

The bulk propagator for positive ρ (or large r) is obtained in an analogous way:

$$
\begin{aligned}
G(x,y)_B &= \int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y}))\int_0^\infty d\omega\ \omega\frac{1}{\vec{k}^2+\omega^2} \\
&\quad \times \rho^{-2}J_2(\omega\rho){\rho'}^{-2}J_2(\omega\rho') \\
&= \int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y}))\frac{1}{\rho_<^2}I_2(k\rho_<)\frac{1}{\rho_>^2}K_2(k\rho_>) \\
&\sim \frac{3}{2\pi^2}\frac{1}{(\vec{x}-\vec{y})^8}, \qquad (4.20)
\end{aligned}
$$

where $\rho_<(\rho_>)$ denotes the smaller (larger) quantity between ρ and ρ'. It is identical to the propagator in the flat 10d spacetime.

In AdS/CFT correspondence in ordinary gauge theory, the following relation plays an important role.

$$
\begin{aligned}
& \int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y}))\rho_<^2 I_2(k\rho_<)\rho_>^2 K_2(k\rho_>) \\
\sim\ & \frac{1}{4}\rho_<^4\delta^4(\vec{x}-\vec{y}), \qquad (4.21)
\end{aligned}
$$

when $\rho_>, \rho_< \to 0$. Hence we can construct the classical solution $\phi(\rho,\vec{x})$ which approaches $\phi(\vec{x})$ as ρ approaches the boundary $\rho_0 \sim 0$:

$$
\begin{aligned}
\phi(\rho,\vec{x}) &= \int d^4y \frac{4}{\rho_0^4}\int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y}))\rho^2 K_2(k\rho)\rho_0^2 I_2(k\rho_0)\phi(\vec{y}) \\
&= \int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot\vec{x})\frac{4}{\rho_0^4}\rho^2 K_2(k\rho)\rho_0^2 I_2(k\rho_0)\phi(\vec{k}). \qquad (4.22)
\end{aligned}
$$

This relation is used to estimate the boundary contribution of the supergravity action. It is a crucial ingredient to reproduce the correlators of local 4d field theory on the boundary. However we are effectively constrained such that $|\rho| > R$ in NC gauge theory. It is because our approximations through Bessel functions are no longer valid when $|\rho| \to R$. We will

verify the existence of the effective cut-off R by an exact analysis through Mathieu functions subsequently. Nevertheless there must be a supergravity description of NC gauge theory which reproduces the correlators in the low momentum regime. We will argue in what follows that such a prescription is to locate the Wilson lines at the maximum of the string metric $r \sim R$ [30].

After the change of the variables as

$$r = Re^{-z}, \varphi = \frac{1}{r^2}\psi(z), \tag{4.23}$$

ψ obeys

$$[-\frac{\partial^2}{\partial z^2} - 2(\omega R)^2 cosh2z]\psi(z) = -4\psi(z). \tag{4.24}$$

(4.24) is identical to the wave equation of scalar fields in the background of the D3-brane metric. Therefore the exact propagator of this problem may be constructed through Mathieu functions. It is interesting to note that we obtain the identical equation of the motion in the large b scaling limit with the D3-brane background without b field. In this coincidence the non-commutative scale in the former plays the role of the string scale in the latter. It is consistent with the proposal that NC gauge theory is a string theory whose string scale is non-commutative scale [41].

The long distance behavior of the propagator is determined by the wave functions with small ω since we will eventually pick a pole at $\omega = ik$. In the small ω limit, our quantum system is separated into two sectors whose wave functions are localized around $z \sim -\infty$ and $z \sim \infty$ due to the large potential energy barrier in the above expression. Therefore our quantum system decouples into two sectors in the low energy limit. It is therefore makes a sense to separate the system into two regimes: $r < R$ regime and $r > R$ regime. It is consistent to locate the Wilson lines at the maximum of the metric $r = R$ with such a separation.

In what follows, we evaluate the supergravity action for a classical solution with the fixed value $\phi(\vec{k})$ at $r = R$. In Appendix (A.11) , we have constructed a classical solution which approaches $\phi(\vec{k})$ at $r = R$ in the small momentum regime. The Wilson lines separate the $r < R$ region from the $r > R$ region. With this classical solution, we can evaluate the contribution from $r < R$ region by the supergravity action (4.1) just like ordinary gauge theory:

$$\frac{\pi^3 R^5}{2g^2} \int \frac{d^4k}{(2\pi)^4} \phi(r, R, \vec{k}) \frac{\partial}{\partial r} \phi(r, R, \vec{k})|_{r=R}$$

$$= \frac{\pi^3 R^8}{2g^2}\int \frac{d^4k}{(2\pi)^4}\phi(\vec{k})\Big(-\frac{1}{8}k^4 log(k^2R^2)\Big)\phi(-\vec{k})$$

$$= \frac{n^2}{16\pi^2}\int \frac{d^4k}{(2\pi)^4}\phi(\vec{k})\Big(-\frac{1}{8}k^4 log(k^2R^2)\Big)\phi(-\vec{k}). \tag{4.25}$$

We can also evaluate the contribution from $r > R$ region:

$$-\frac{\pi^3 R^5}{2g^2}\int \frac{d^4k}{(2\pi)^4}\phi(R,r',\vec{k})\frac{\partial}{\partial r'}\phi(R,r',\vec{k})|_{r'=R}$$

$$= \frac{\pi^3 R^8}{2g^2}\int \frac{d^4k}{(2\pi)^4}\phi(\vec{k})\Big(-\frac{1}{8}k^4 log(k^2R^2)\Big)\phi(-\vec{k})$$

$$= \frac{n^2}{16\pi^2}\int \frac{d^4k}{(2\pi)^4}\phi(\vec{k})\Big(-\frac{1}{8}k^4 log(k^2R^2)\Big)\phi(-\vec{k}). \tag{4.26}$$

The sign difference between the above two expressions originates from the fact that the former picks up the upper boundary contribution while the latter picks up the lower boundary contribution with respect to r integration. The both give the identical contributions to the two point function in agreement with the propagator itself at $r = R$ which is evaluated in Appendix (A.10).

We recall here that the two point functions of the Wilson lines receive contributions from the planar and non-planar sectors. The non-analytic behavior arises not only from the small but also from large momentum contributions due to UV/IR mixing in the non-planar sector. The both contributions result in the identical non-analytic behavior of the correlators leading to the identical long range interaction. The planar contributions are identical to ordinary gauge theory while the non-planar contributions are of the same magnitude. In supergravity, we have also two decoupled sectors in the small momentum limit: $r < R$ and $r > R$ regimes. We find that the supergravity action in these two different regimes can account for the identical non-analytic behaviors. It is reassuring that we can reproduce these essential features of the Wilson line correlators in NC gauge theory from supergravity. It a posteriori justifies our interpreting the contributions from $r < R$ and $r > R$ regimes in supergravity as the IR and UV/IR mixing contributions in NC gauge theory respectively. In fact the non-analytic part of the propagator near $r = R$ (A.10) is the sum of (4.18) for $r < R$ and (4.20) for $r > R$ regimes. Our prescription in supergravity is successful to describe this important feature of two point functions of NC gauge theory.

5 Conclusions and Discussions

In this paper, we have investigated the two point correlation functions of the Wilson lines in NC gauge theory. We have focused on $\mathcal{N} = 4$ gauge theory on $S^2 \times S^2$ which is realized by IIB matrix model. We have found finite quantum corrections to the non-commutative extension of BPS operators which carry finite momenta. We have further given a perturbative prescription to obtain local operators with no anomalous dimensions in the small momentum regime. We have argued that our prescription is legitimate as it removes one loop quantum corrections of all correlators since the renormalization effect is associated with individual operators.

It has been conjectured that these correlators are described by dual supergravity in the strong 't Hooft coupling regime. Our findings in this paper summarized above support such a conjecture. We find extra contributions to the correlators due to the UV/IR mixing effect in addition to the identical IR contributions with ordinary gauge theory. We can successfully reproduce these characteristic features of NC gauge theory by locating the Wilson lines at the maximum of the string metric in dual supergravity description.

In AdS/CFT correspondence, ordinary 4d gauge theory is located at the boundary of AdS_5 where the metric diverges. Since an arbitrary small distance in field theory corresponds to a finite physical distance in such a situation, it is consistent to propose that a field theory holographically realizes supergravity and string theory. However the metric does not diverge anywhere in the supergravity background (4.3) which is relevant to NC gauge theory. This feature is consistent with the fact that the both NC gauge theory and string theory have minimum length scale. By locating the NC gauge theory at the maximum of the string metric, we can probe the shortest length scale in NC gauge theory for a fixed supergravity length scale . In more generic spacetime such as flat or de Sitter spacetime, the metric does not diverge either. We thus believe that supergravity background dual to NC gauge theory has much in common with physically realistic spacetime.

In the one loop effective action of NC gauge theory constructed from IIB matrix model, the bilinear terms of the Wilson lines appear due to the non-planar diagrams [26]. The leading terms are of the same type with (4.26) after replacing $n\phi$ by the Wilson lines. Since such terms arise due to graviton exchange in the bulk, they constitute an evidence for the existence of dynamical supergravity in NC gauge theory with finite n. It is thus conceivable that non-planar sectors may be effectively described by dynamical supergravity.

Such a possibility is certainly the most attractive feature of matrix models. We would like to extend our scope of research to such a problem by investigating finite n corrections.

We can further mention that $\varphi = 1$ is a solution of (4.24) with $\omega = 0$. It has been pointed out that this zeromode could give rise to $4d$ gravity a la Randall and Sundrum [41]. We have shown that it makes sense to divide supergravity action into two sectors: $r < R$ and $r > R$ regimes. Let us substitute $\varphi = \varphi(\vec{x})$ into the supergravity action in the $r < R$ regime. We obtain the following contribution:

$$\frac{1}{8g^2}\int_0^R drr^5 d^4x\partial_i\varphi(\vec{x})\partial_i\varphi(\vec{x})(1+\frac{R^4}{r^4}) = \frac{3R^6}{16g^2}\int d^4x\partial_i\varphi(\vec{x})\partial_i\varphi(\vec{x}), \tag{5.1}$$

which gives rise to massless fields in 4 dimensions. Suppose $\varphi(\vec{x})$ is coupled to energy momentum tensor T at $r = R$ as

$$\int d^4xT(\vec{x})\varphi(\vec{x}). \tag{5.2}$$

We then obtain 4d gravity with Newton's law whose gravitational coupling constant is R^2/n^2. However we have not found the direct evidence for this phenomenon in our perturbative analysis in this paper. Presumably this very interesting possibility may be realized through non-perturbative effects in NC gauge theory.

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Appendix A

In this Appendix, we construct a classical solution in the bulk which approaches $\phi(\vec{k})$ at $r = R$ in the small momentum regime by Mathieu functions [42] [3]. The two independent solutions of (4.24) can be chosen to be the Floquet solutions:

$$J(\nu, z),\ \ J(-\nu, z). \tag{A.1}$$

Here the parameter ν is determined in terms of the combination ωR. It has a power series expansion given by

$$\nu = 2 - \frac{i\sqrt{5}}{3}(\frac{\omega R}{2})^4 + \frac{7i}{108\sqrt{5}}(\frac{\omega R}{2})^8 + \cdots. \tag{A.2}$$

[3] We use the identical notations with that reference.

For our purpose, it is more appropriate to consider

$$\begin{aligned} H^{(1)}(\nu,z) &= \frac{2}{C}\Big(J(-\nu,z)-\frac{1}{\eta}J(\nu,z)\Big) \\ H^{(2)}(\nu,z) &= -\frac{2}{C}\Big(J(-\nu,z)-\eta J(\nu,z)\Big), \end{aligned} \tag{A.3}$$

where $\eta = exp(i\pi\nu)$ and $C = \eta - 1/\eta$. The Mathieu functions approach respective Bessel functions as $Rez \to \infty$:

$$Z^{(j)}(\nu,z) \to Z^{(j)}_{\nu}(\sqrt{q}e^{z}), \tag{A.4}$$

where Z denotes J or H.

The exact propagator due to excited states has been constructed in [43]

$$\begin{aligned} G(x,y)_B &= \int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y})) \\ & (\frac{1}{rr'})^2 \frac{C}{2A} H^{(2)}(\nu,-z'-\frac{i\pi}{2})|_{\omega=k} \frac{\pi}{2i} H^{(1)}(\nu,z+\frac{i\pi}{2})|_{\omega=k}. \end{aligned} \tag{A.5}$$

In the small r, r' regime ($z > z' > 0$), we can evaluate it as

$$\begin{aligned} G(x,y)_B &= \int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y})) \\ & (\frac{1}{rr'})^2 \Big(-J(\nu,z'+\frac{i\pi}{2}) + \frac{\eta C}{2A\chi} H^{(1)}(\nu,z'+\frac{i\pi}{2})\Big)|_{\omega=k} \\ & \times \frac{\pi}{2i} H^{(1)}(\nu,z+\frac{i\pi}{2})|_{\omega=k}. \end{aligned} \tag{A.6}$$

In the large r, r' regime ($-z' > -z > 0$), it can be evaluated as

$$\begin{aligned} G(x,y)_B &= \int \frac{d^4k}{(2\pi)^4} exp(i\vec{k}\cdot(\vec{x}-\vec{y})) \\ & \times(\frac{1}{rr'})^2 \Big(J(\nu,-z-\frac{i\pi}{2}) + \frac{C}{2A\chi\eta} H^{(2)}(\nu,-z-\frac{i\pi}{2})\Big)|_{\omega=k} \\ & \times \frac{\pi}{2i} H^{(2)}(\nu,-z'-\frac{i\pi}{2})|_{\omega=k}. \end{aligned} \tag{A.7}$$

From the asymptotic behaviors in (A.4), we can see that this propagator indeed approaches (4.18) when $z, z' \to \infty$ and (4.20) when $z, z' \to -\infty$.

The long range behavior of the propagator can be estimated by its small momentum behavior. In such a limit, the following quantities behave as

$$\begin{aligned} &\chi \sim (-\frac{2}{3} - i\frac{\sqrt{5}}{3})\Big(1 + i\frac{2\sqrt{5}}{3}(\frac{kR}{2})^4 log(\frac{kR}{2})\Big) = \chi_0 + O(k^4), \\ &A \equiv \chi - \frac{1}{\chi} = -2i\frac{\sqrt{5}}{3}\Big(1 - (\chi_0 + \frac{1}{\chi_0})(\frac{kR}{2})^4 log(\frac{kR}{2})\Big) = A_0 + O(k^4), \\ &\eta \sim 1 + \frac{\pi\sqrt{5}}{3}(\frac{kR}{2})^4,\ C = \eta - \frac{1}{\eta} \sim \frac{2\pi\sqrt{5}}{3}(\frac{kR}{2})^4. \end{aligned} \tag{A.8}$$

The Mathieu functions can be expanded as

$$
\begin{aligned}
& J(\nu, z+\frac{i\pi}{2})|_{\omega=k} \\
\sim\ & -\frac{1}{2}(\frac{kR}{2})^2(\frac{R^2}{r^2}+\frac{1}{\chi_0}\frac{r^2}{R^2})\Big(1-i\frac{\sqrt{5}}{3}(\frac{kR}{2})^4 log(\frac{kR}{2}\frac{R}{r})\Big), \\
& J(-\nu, z+\frac{i\pi}{2})|_{\omega=k} \\
\sim\ & -\frac{1}{2}(\frac{kR}{2})^2(\frac{R^2}{r^2}+\chi_0\frac{r^2}{R^2})\Big(1+i\frac{\sqrt{5}}{3}(\frac{kR}{2})^4 log(\frac{kR}{2}\frac{R}{r})\Big), \\
& H^{(1)}(\nu, z+\frac{i\pi}{2})|_{\omega=k} \\
\sim\ & -\frac{A_0}{C}(\frac{kR}{2})^2\frac{r^2}{R^2}\Big(1-\frac{1}{2}(\frac{kR}{2})^4\Big(2\frac{R^4}{r^4}+(\chi_0+\frac{1}{\chi_0})\Big)log(\frac{kR}{2}\frac{R}{r})\Big), \\
& H^{(2)}(\nu, -z-\frac{i\pi}{2})|_{\omega=k} \\
\sim\ & \frac{A_0}{C}(\frac{kR}{2})^2\frac{R^2}{r^2}\Big(1-\frac{1}{2}(\frac{kR}{2})^4\Big(2\frac{r^4}{R^4}+(\chi_0+\frac{1}{\chi_0})\Big)log(\frac{kR}{2}\frac{r}{R})\Big).
\end{aligned} \tag{A.9}
$$

In this way we find the integrand of the propagator in (A.5) behaves as

$$
\begin{aligned}
& (\frac{1}{rr'})^2\frac{C}{2A}H^{(2)}(\nu,-z'-\frac{i\pi}{2})|_{\omega=k}\frac{\pi}{2i}H^{(1)}(\nu,z+\frac{i\pi}{2})|_{\omega=k} \\
& \sim \frac{1}{4}(\frac{1}{r'})^4\Big(1-(\frac{kR}{2})^4(\frac{r'^4}{R^4}+\frac{R^4}{r^4})log(\frac{kR}{2})\Big).
\end{aligned} \tag{A.10}
$$

We can also determine the small momentum expansion of the classical solutions as follows

$$
\begin{aligned}
\phi(r,R,\vec{k}) &\sim \Big(1-(\frac{kR}{2})^4(\frac{R^4}{r^4}-1)log(\frac{kR}{2})\Big)\phi(\vec{k}),\ (r<R), \\
\phi(R,r',\vec{k}) &\sim \frac{R^4}{r'^4}\Big(1-(\frac{kR}{2})^4(\frac{r'^4}{R^4}-1)log(\frac{kR}{2})\Big)\phi(\vec{k}),\ (R<r'),
\end{aligned} \tag{A.11}
$$

where the overall k dependent normalization is fixed by the requirement that they coincide $\phi(\vec{k})$ when $r, r' \to R$.

The propagator in (A.10) vanishes in the both asymptotic regions when $r \to 0$ or $r' \to \infty$. When we locate the Wilson lines at $r = R$, such a requirement may be too restrictive. For example, the following term can be added to it in the $r, r' > R$ regime with an arbitrary coefficient since it solves the equation of motion:

$$
(\frac{1}{rr'})^2\frac{\pi}{2i}H^{(2)}(\nu,-z'-\frac{i\pi}{2})|_{\omega=k}\frac{\pi}{2i}H^{(2)}(\nu,-z-\frac{i\pi}{2})|_{\omega=k} \sim \frac{1}{4}(\frac{R}{rr'})^4(\frac{2}{kR})^4. \tag{A.12}
$$

Such a non-analytic behavior may explain the non-planar contributions to the two point correlators of more generic Wilson line operators we have mentioned in (2.44).

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